17 Self-Adjoint and Normal Operators

17.1 Adjoints

Adjoint, T^* Suppose $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. The **adjoint** of T is the function $T^* : \mathcal{W} \to \mathcal{V}$ such that

$$\langle Tv, w\rangle = \langle v, T^*w\rangle$$

for every $v \in \mathcal{V}$ and every $w \in \mathcal{W}$.

1. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$$

Find a formula for T^*.

2. Fix $u \in \mathcal{V}$ and $x \in \mathcal{W}$. Define $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ by

$$Tv = \langle v, u\rangle x$$

for every $v \in \mathcal{V}$. Find a formula for T^*.

The adjoint is a linear map
If $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, then $T^* \in \mathcal{L}(\mathcal{W}, \mathcal{V})$.

Properties of the adjoint

(a) $(S + T)^* = S^* + T^*$ for all $S, T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

(b) $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

(c) $(T^*)^* = T$ for all $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

(d) $I^* = I$, where I is the identity operator on $\mathcal{V}$.

(e) $(ST)^* = T^*S^*$ for all $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $S \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ (here $\mathcal{U}$ is an inner product space over $\mathbb{F}$).

Null space and range of T^*
Suppose $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Then

(a) $\ker T^* = (\operatorname{im} T)^\perp$.

(b) $\operatorname{im} T^* = (\ker T)^\perp$.

(c) $\ker T = (\operatorname{im} T^*)^\perp$.

(d) $\operatorname{im} T = (\ker T^*)^\perp$.

Conjugate transpose The **conjugate transpose** of an m-by-n matrix is the n-by-m matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

The matrix of T^* Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Suppose $\mathcal{B} = \{e_1, ..., e_n\}$ is an orthonormal basis of $\mathcal{V}$ and $\mathcal{B}' = \{f_1, ..., f_m\}$ is an orthonormal basis of $\mathcal{W}$. Then

$$[T^*]_{\mathcal{B}'\mathcal{B}}$$

is the conjugate transpose of

$$[T]_{\mathcal{B}\mathcal{B}'}.$$

17.2 Self-Adjoint Operators

Self-Adjoint (or Hermitian) Operator
An operator $T \in \mathcal{L}(\mathcal{V})$ is called **self-adjoint** if $T = T^*$. In other words, $T \in \mathcal{L}(\mathcal{V})$ is self-adjoint if and only if

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

for all $v, w \in \mathcal{V}$.

3. Suppose T is the operator on $\mathbb{F}^2$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 2 & b \\ 3 & 7 \end{pmatrix}.$$

Find all numbers b such that T is self-adjoint.

Eigenvalues of self-adjoint operators are real
Every eigenvalue of a self-adjoint operator is real.

Over $\mathbb{C}$, Tv is orthogonal to v for all v only for the 0 operator
Suppose $\mathcal{V}$ is a complex inner product space and $T \in \mathcal{L}(\mathcal{V})$. Suppose

$$\langle Tv, v\rangle = 0$$

for all $v \in \mathcal{V}$. Then $T = 0$.

Over $\mathbb{C}$, $\langle Tv, v\rangle$ is real for all v only for self-adjoint operators
Suppose $\mathcal{V}$ is a complex inner product space and $T \in \mathcal{L}(\mathcal{V})$. Then T is self-adjoint if and only if

$$\langle Tv, v\rangle \in \mathbb{R}$$

for every $v \in \mathcal{V}$.

If $T = T^*$ and $\langle Tv, v\rangle = 0$ for all v, then $T = 0$
Suppose T is a self-adjoint operator on $\mathcal{V}$ such that

$$\langle Tv, v\rangle = 0$$

for all $v \in \mathcal{V}$. Then $T = 0$.

17.3 Normal operators

Normal Operator

- An operator on an inner product space is called **normal** if it commutes with its adjoint.
- In other words, $T \in \mathcal{L}(\mathcal{V})$ is normal if

$$TT^* = T^*T.$$

4. Let T be the operator on $\mathbb{F}^2$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

Show that T is not self-adjoint and that T is normal.

T is normal if and only if $\|Tv\| = \|T^*v\|$ for all v

An operator $T \in \mathcal{L}(\mathcal{V})$ is normal if and only if

$$\|Tv\| = \|T^*v\|$$

for all $v \in \mathcal{V}$.

For T normal, T and T^* have the same eigenvectors

Suppose $T \in \mathcal{L}(\mathcal{V})$ is normal and $v \in \mathcal{V}$ is an eigenvector of T with eigenvalue λ. Then v is also an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

Orthogonal eigenvectors for normal operators

Suppose $T \in \mathcal{L}(\mathcal{V})$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

17.4 Exercises

5. Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, ..., z_n) = (0, z_1, ..., z_{n-1}).$$

Find a formula for $T^*(z_1, ..., z_n)$.

6. Suppose $T \in \mathcal{L}(\mathcal{V})$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

7. Suppose $T \in \mathcal{L}(\mathcal{V})$ and $\mathcal{U}$ is a subspace of $\mathcal{V}$. Prove that $\mathcal{U}$ is invariant under T if and only if $U^{\perp}$ is invariant under T^*.

8. Suppose $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Prove that

(a) T is injective if and only if T^* is surjective;

(b) T is surjective if and only if T^* is injective.

9. Prove that

$$\dim \ker T^* = \dim \ker T + \dim W - \dim V$$

and

$$\dim \operatorname{im} T = \dim \operatorname{im} T$$

for every $T \in \mathcal{L}(V, W)$.

10. Make $P_2(\mathbb{R})$ into an inner product space by defining

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x.$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

(a) Show that T is not self-adjoint.

(b) The matrix of T with respect to the basis $\{1, x, x^2\}$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

11. Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

12. Suppose V is a real inner product space. Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

13. Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

14. Suppose $\dim V \geq 2$. Show that the set of normal operators on V is not a subspace of $L(V)$.

15. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self-adjoint.

16. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

17. Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ such that T is normal but not self-adjoint.

18. Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \quad Tv = 3, \quad Tw = 4w.$$

Show that $\|T(v + w)\| = 10$.